2021-2022 AUT Admission Examination

Mathematics

Solution to SAMPLE Exam



Test ID Number	
Full Name	
Major	

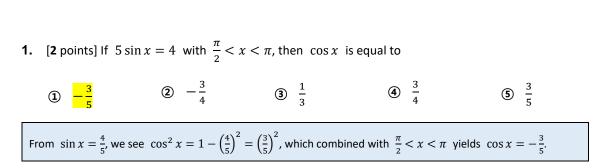


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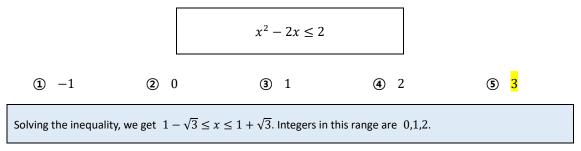
(Mathematics)

Sample

Multiple choice questions



2. [2 points] Find the sum of all **INTEGERS** x satisfying the following inequality:



3. [2 points] Which of the following is the LARGEST?

(1)
$$\sqrt{3}$$
 (2) $\sqrt[3]{3\sqrt{2}}$ (3) $\sqrt{2\sqrt[3]{3}}$ (4) $\sqrt[3]{5}$ (5) $\sqrt[6]{23}$
From $\sqrt{3} = \sqrt[6]{27}$, $\sqrt[3]{3\sqrt{2}} = \sqrt[6]{18}$, $\sqrt{2\sqrt[3]{3}} = \sqrt[6]{24}$, $\sqrt[3]{5} = \sqrt[6]{25}$, we see $\sqrt{3}$ is the largest.

4. [3 points] If $\omega^2 - \omega + 1 = 0$, then $\omega^{2021} - 2\omega^{2022} + 3\omega^{2023}$ is equal to

(1) 1 (2) $2\omega - 1$ (3) $2\omega + 1$ (4) ω^2 (5) $2\omega^2$

Multiplying $(\omega + 1)$ on both sides of $\omega^2 - \omega + 1 = 0$, we obtain $\omega^3 + 1 = 0$, i.e. $\omega^3 = -1$. So, $\omega^{2021} - 2\omega^{2022} + 3\omega^{2023} = (\omega^3)^{673}\omega^2 - 2(\omega^3)^{674} + 3(\omega^3)^{674}\omega$ $= -\omega^2 - 2 + 3\omega = (1 - \omega) - 2 + 3\omega = 2\omega - 1$

5. [**3** points] If α and β are the roots of the equation $(\log_3 x)^2 - \log_3 x^3 = 9$, then $\alpha\beta$ is equal to

 1)
 27
 2)
 18
 3)
 9
 4)
 6
 5)
 2

Observe that $\log_3 \alpha$ and $\log_3 \beta$ are the roots of the equation	$t^2 - 3t - 9 = 0$. By properties of quadratic
equations, $\log_3 \alpha + \log_3 \beta = 3$, and so $\log_3(\alpha\beta) = 3$. Therefore	, $\alpha\beta = 27$.

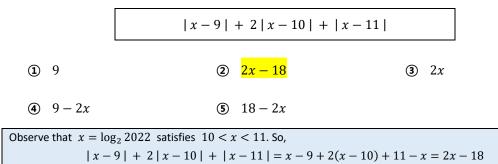


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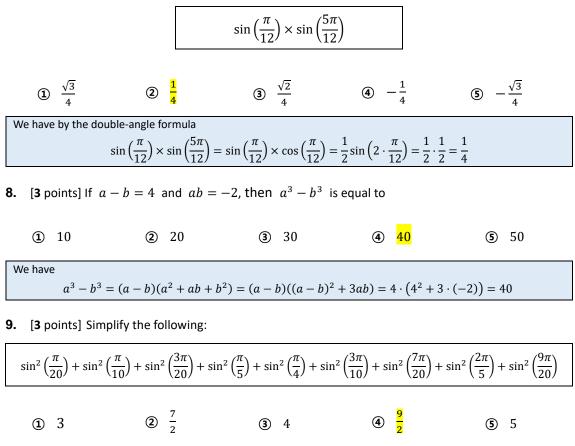
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6. [**3** points] Let $2^x = 2022$. Simplify the following:



7. [3 points] Simplify the following:



From

$$\sin\left(\frac{9\pi}{20}\right) = \cos\left(\frac{\pi}{20}\right), \sin\left(\frac{2\pi}{5}\right) = \cos\left(\frac{\pi}{10}\right), \sin\left(\frac{7\pi}{20}\right) = \cos\left(\frac{3\pi}{20}\right), \sin\left(\frac{3\pi}{10}\right) = \cos\left(\frac{\pi}{5}\right), \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
 ind the square identity, we see the given expression is equal to 9/2.

10. [**3** points] Simplify the following:

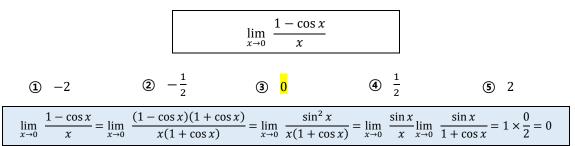
$$\boxed{ \log_7 \left(1 - \frac{1}{2}\right) + \log_7 \left(1 - \frac{1}{3}\right) + \log_7 \left(1 - \frac{1}{4}\right) + \dots + \log_7 \left(1 - \frac{1}{49}\right) }$$

$$\boxed{ (1 - 4 \quad (2) \quad -2 \quad (3 \quad 0 \quad (4 \quad 2 \quad (5 \quad 4)) }$$
By the properties of the logarithmic functions, we get
$$\log_7 \left(1 - \frac{1}{2}\right) + \log_7 \left(1 - \frac{1}{3}\right) + \log_7 \left(1 - \frac{1}{4}\right) + \dots + \log_7 \left(1 - \frac{1}{49}\right) = \log_7 \left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{48}{49}\right) = \log_7 \frac{1}{49} = -2$$

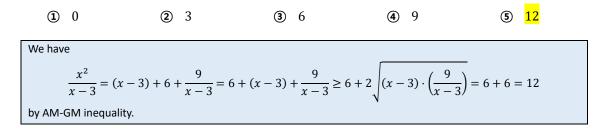


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11. [**3** points] Evaluate the following limit:



12. [**3** points] Find the <u>MINIMUM</u> value of $\frac{x^2}{x-3}$ for x > 3.



13. [**3** points] Let
$$A = \begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}$$
. If $A^2 = \begin{pmatrix} b & 9 \\ c & d \end{pmatrix}$. Then, $a + b + c + d$ is equal to

 2 	2 4	36	4 8	(5) 10
We have and so $a = 3, b =$	$A^2 = \begin{pmatrix} 1\\ 0 \\ 1, \ c = 0, \text{ and } d = 4.$	$ \begin{pmatrix} a \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} $ Therefore, $a + b + c$		

14. [3 points] A differentiable function f(x) defined on the real line has the following values:

x	-1	0	2	
f(x)	1	7	3	
f'(x)	4	1	-2	

Find g'(1) for $g(x) = (f(2x))^3$.

1	<mark>–108</mark>	2 -36	③ −12	④ 12	⑤ 36

By the chain rule, we have
$g'(x) = 3(f(2x))^2 \cdot 2f'(2x)$
and so $g'(1) = 6(f(2))^2 f'(2) = 6 \cdot 3^2 \cdot (-2) = -108$ with the aid of the table.



15. [**3** points] Suppose the following holds for some real numbers *a* and *b*.

$$\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)^{161} = a + b i$$

Then, ab is equal to

(1)
$$-\frac{1}{2}$$
 (2) $-\frac{1}{\sqrt{2}}$ (3) $\frac{\sqrt{3}}{4}$ (4) $\frac{1}{2}$ (5) $\frac{1}{\sqrt{2}}$

We have

$$\omega \coloneqq \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Simple calculation shows $\omega^2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $\omega^3 = i$. Then, $\omega^{161} = (\omega^3)^{52+1}\omega^2 = i\omega^2 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. This shows $a = \frac{\sqrt{3}}{2}$ and $b = \frac{1}{2}$, and so $ab = \frac{\sqrt{3}}{4}$.

16. [3 points] If
$$\mathbb{P}(A|B) = \frac{1}{5}$$
, $\mathbb{P}(B|A) = \frac{1}{2}$, and $\mathbb{P}(A \cup B) = \frac{1}{4}$, find $\mathbb{P}(A \cap B)$.

(1)
$$\frac{1}{6}$$
 (2) $\frac{1}{12}$ (3) $\frac{1}{15}$ (4) $\frac{1}{18}$ (5) $\frac{1}{24}$

we have

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{5}, \qquad \mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{1}{2}$$
and so $\mathbb{P}(A) = 2\mathbb{P}(A \cap B)$ and $\mathbb{P}(B) = 5\mathbb{P}(A \cap B)$. From
 $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 6\mathbb{P}(A \cap B) = \frac{1}{4}$
we get $\mathbb{P}(A \cap B) = \frac{1}{24}$.

17. [**3** points] Find the **MINIMUM** value of x + 2y where $x^2 + y^2 = 1$.

(1)
$$-2$$
 (2) $-\sqrt{5}$ (3) $-\sqrt{3}$ (4) $-\sqrt{2}$ (5) -1
Let
 $x + 2y = k$
Then, $(k - 2y)^2 + y^2 = 1$ has real roots. So, the discriminant of $5y^2 - 4ky + k^2 - 1 = 0$ is nonnegative:
 $4k^2 - 5(k^2 - 1) \ge 0$
In other words, $k^2 \le 5$. So, the minimum value of k is $-\sqrt{5}$.



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18. [**3** points] Suppose that a differentiable function f satisfies

$$f(x) = \sin x + \int_0^{\pi} (f'(t))^2 dt$$

for all x. Then, $f(\pi)$ is equal to

(1)
$$\frac{\pi}{3}$$
 (2) $\frac{\pi}{2}$ (3) π (4) 0 (5) 1
Let $\int_{-\pi}^{\pi} (f'(t))^2 dt = k$

Then,
$$f(x) = \sin x + k$$
. So,
 $k = \int_0^{\pi} (f'(t))^2 dt = \int_0^{\pi} (\cos t)^2 dt = \int_0^{\pi} \frac{1}{2} (1 + \cos 2t) dt = \frac{\pi}{2}$
 $f(\pi) = \sin \pi + \frac{\pi}{2} = \frac{\pi}{2}$

19. [**3** points] A quadratic function y = f(x) satisfies f(0) = 1 and

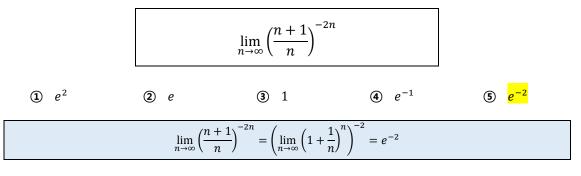
$$\int_{-1}^{2} f(x)dx = \int_{-1}^{0} f(x)dx = \int_{0}^{2} f(x)dx$$

Then, f(-2) is equal to

(1)
$$-7$$
 (2) -6 (3) -5 (4) -4 (5) -3
Let $f(x) = ax^2 + bx + 1$. From the conditions, we see
$$\int_{-1}^{0} f(x)dx = \int_{0}^{2} f(x)dx = 0$$

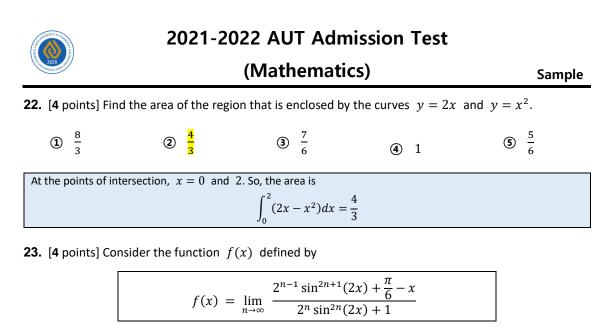
This gives $a = -3/2$ and $b = 1$. So, $f(x) = -\frac{3}{2}x^2 + x + 1$. Hence, $f(-2) = \left(-\frac{3}{2}\right)(-2)^2 + (-2) + 1 = -7$.

20. [**3** points] Evaluate the following limit:



21. [4 points] Find the area of the triangle \triangle ABC with sides $\overline{AB} = 7$, $\overline{BC} = 4$, and $\overline{AC} = 5$.

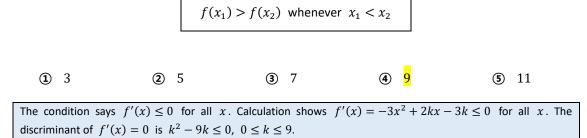
(1)
$$2\sqrt{3}$$
 (2) $2\sqrt{6}$ (3) $2\sqrt{7}$ (4) $4\sqrt{6}$ (5) $4\sqrt{7}$
By the second law of cosine,
 $\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{7^2 + 4^2 - 5^2}{2 \cdot 7 \cdot 4} = \frac{5}{7}$
and so $\sin B = \frac{2\sqrt{6}}{7}$. The area is $\frac{1}{2}ac \sin B = \frac{1}{2} \cdot 7 \cdot 4 \cdot \frac{2\sqrt{6}}{7} = 4\sqrt{6}$



Then, $(f \circ f)(0)$ is equal to

(1) 0 (2)
$$\frac{\sqrt{3}}{4}$$
 (3) $\frac{\sqrt{3}}{2}$ (4) $\frac{1}{2\sqrt{2}}$ (5) $\frac{1}{4\sqrt{2}}$
We can easily see that $f(0) = \frac{\pi}{6}$. Since $\left|2\sin^2\frac{\pi}{3}\right| > 1$, we obtain $f(f(0)) = f\left(\frac{\pi}{6}\right) = \frac{1}{2}\sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}$

24. [4 points] Find the **LARGEST** real number k such that $f(x) = -x^3 + kx^2 - 3kx + 1$ satisfies



25. [4 points] Suppose the function defined by

$$f(x) = \int_0^x e^{t^2} dt$$

satisfies $f(a) = \frac{\pi}{2}$ for some constant *a*. Then,

$$\int_0^a \sin(f(x)) e^{x^2} dx$$

is equal to

(1)
$$\frac{1}{\pi}$$
 (2) $\frac{1}{2}$ (3) (4) 2 (5) π

With substitution $u = \int_0^x e^{t^2} dt$, $du = e^{x^2} dx$, we obtain

$$\int_{0}^{a} \sin(f(x)) e^{x^{2}} dx = \int_{0}^{\frac{\pi}{2}} \sin u \, du = 1$$



(Mathematics)

Sample

Short answer questions

26. [4 points] Find the sum of all x with $5 \le x \le 500$ such that $\log_{10} x$ is an integer.

x = 10, 100

Answer: 110

27. [4 points] Evaluate the following integral:

$$\int_1^4 x \sqrt{17 - x^2} dx$$

$$u = 17 - x^{2}, du = -2xdx \text{ gives}$$
$$\int_{1}^{4} x \sqrt{17 - x^{2}} dx = \int_{1}^{16} \frac{1}{2} \sqrt{u} du = \frac{1}{3} u \sqrt{u} \Big|_{1}^{16} = \frac{1}{3} (64 - 1) = 21$$

Answer: 21

28. [5 points] Let α be the sum of <u>ALL</u> solutions to the trigonometric equation

$$\cos 2x - \cos x + 1 = 0, \qquad 0 \le x \le \pi.$$

Evaluate



With $t = \cos x$, the equation becomes $2t^2 - 1 - t + 1 = 0$ by double-angle formula. Then, t = 0 or t = 1/2. Corresponding x values are $\frac{\pi}{2}$ and $\frac{\pi}{3}$.

Answer: 60

29. [5 points] Find the real number k such that the equation

$$\frac{(\ln x)^4}{x} = e^{-4}k$$

has **<u>TWO DISTINCT</u>** real roots.

Let $f(x) = \frac{(\ln x)^4}{x}$. Then, from $f'(x) = \frac{(\ln x)^3(4-\ln x)}{x^2}$, we see that f(x) has a local minimum f(1) = 0 and local maximum $f(e^4) = \frac{256}{e^4}$. Also, we have $\lim_{x \to \infty} f(x) = 0$. Therefore, the given equation has two distinct real roots when k = 256.



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30. [5 points] Evaluate the following limit:

$$\lim_{n \to \infty} \sqrt[3]{n} \left(\sqrt[3]{n^2 + 2022n + 1} - \sqrt[3]{n^2 + 1} \right)$$

We have

$$\sqrt[3]{n}\left(\sqrt[3]{n^2 + 2022n + 1} - \sqrt[3]{n^2 + 1}\right) = \frac{\sqrt[3]{n}\left(n^2 + 2022n + 1 - (n^2 + 1)\right)}{\left(\sqrt[3]{n^2 + 2022n + 1}\right)^2 + \sqrt[3]{n^2 + 2022n + 1}\sqrt[3]{n^2 + 1} + \left(\sqrt[3]{n^2 + 1}\right)^2}$$
$$= \frac{2022\sqrt[3]{n}n}{\left(\sqrt[3]{n^2 + 2022n + 1}\right)^2 + \sqrt[3]{n^2 + 2022n + 1}\sqrt[3]{n^2 + 1} + \left(\sqrt[3]{n^2 + 1}\right)^2}$$
$$\to \frac{2022}{3} = 674$$

Answer: 674